A class function on the mapping class group of an orientable surface and the Meyer cocycle

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Abstract

In this paper we define a $\mathbf{QP}^1$-valued class function on the mapping class group $\mathcal{M}_{g,2}$ of a surface $\Sigma_{g,2}$ of genus g with two boundary components. Let E be a $\Sigma_{g,2}$ bundle over a pair of pants P. Gluing to E the product of an annulus and P along the boundaries of each fiber, we obtain a closed surface bundle over P. We have another closed surface bundle by gluing to E the product of P and two disks.

The sign of our class function cobounds the 2-cocycle on $\mathcal{M}_{g,2}$ defined by the difference of the signature of these two surface bundles over P.

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0 Introduction

Let $\Sigma_{g,r}$ be a compact oriented surface of genus g with r boundary components. The mapping class group $\mathcal{M}_{g,r}$ is $\pi_0 \operatorname{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ where $\operatorname{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ is the group of orientation preserving diffeomorphisms of $\Sigma_{g,r}$ which restrict to the identity on the boundary $\partial\Sigma_{g,r}$. We simply denote $\Sigma_g := \Sigma_{g,0}$ and $\mathcal{M}_g := \mathcal{M}_{g,0}$. Harer[4] proved that

$$H^2(\mathcal{M}_{g,r}; \mathbf{Z}) \cong \mathbf{Z} \quad g \geq 3,\ r \geq 0,$$

see also Korkmaz, Stipsicz[8]. Meyer[9] defined a cocycle $\tau_g \in Z^2(\mathcal{M}_g; \mathbf{Z})$ $(g \geq 0)$ called the Meyer cocycle

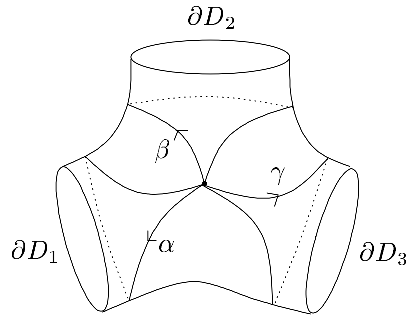

Figure 1:

which represents four times generator of the second cohomology class when $g \geq 3$. Let $P := S^2 - \amalg_{i=1}^3 \operatorname{Int} D_i$ where $D_i \subset S^2$ is a disk, $\operatorname{Int} D_i$ its interior in S^2, and $\alpha, \beta, \gamma \in \pi_1(P)$ be the homotopy classes as shown in Figure 1. We consider a $\Sigma_{g,r}$ bundle $E_{g,r}^{\varphi,\psi}$ on the pair of pants P which has monodromies φ, ψ, $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along α, β, $\gamma \in \pi_1(P)$. The diffeomorphism type of $E_{g,r}^{\varphi,\psi}$ does not depend on the choice of representatives in the mapping classes φ and ψ. The Meyer cocycle is defined by

$$\begin{array}{rccccc} \tau_g : & \mathcal{M}_g & \times & \mathcal{M}_g & \rightarrow & \mathbf{Z}\,, \\ & (\ \varphi & , & \psi\ ) & \mapsto & \operatorname{Sign} E_g^{\varphi,\psi} \end{array}$$

where $\operatorname{Sign} E_g^{\varphi,\psi}$ is the signature of the compact 4-manifold $E_g^{\varphi,\psi}$. For $k > 0$, it is known as Novikov additivity that when two compact oriented $4k$-manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signature. When a pants decomposition of a closed 2-manifold is given, the signature of a Σ_g bundle on the 2-manifold is the sum of the signature of the σ_g bundles restricted to each pair of pants. Therefore, it is important to study the Meyer cocycle to calculate the signature of compact 4-manifolds. For $g = 1, 2$ the Meyer cocycle τ_g is a coboundary, and the cobounding function of this cocycle is calculated by several authors, for instance, Meyer[9], Atiyah[1], Kasagawa[6], Iida[5]. The Meyer cocycle is not a coboundary if genus $g \geq 3$, but the cocycle can be a coboundary when it is restricted to a certain subgroup, and calculated by Endo[2], Morifuji[10].

Let I be the unit interval $[0,1] \subset \mathbf{R}$. By sewing a pair of disks onto the surface $\Sigma_{g,2}$ along the boundary, we have Σ_g. For $h \in \operatorname{Diff}_+(\Sigma_{g,2}, \partial\Sigma_{g,2})$, if we extend h by the identity on the pair of disks, we have a self-

diffeomorphism of Σ_g. we denote it $h \cup id_{\amalg_{i=1}^2 D^2}$. By sewing an annulus $S^1 \times I$ onto the surface $\Sigma_{g,2}$ along the boundary, we have Σ_{g+1}. In the same way, if we extend $h \in \mathrm{Diff}_+(\Sigma_{g,2}, \partial\Sigma_{g,2})$ by the identity on the annulus, we have a self-diffeomorphism $h \cup id_{S^1 \times I}$.

Define the induced homomorphism on the mapping class group by

$$\begin{array}{cccc} \theta: & \mathcal{M}_{g,2} & \to & \mathcal{M}_g \\ & [h] & \mapsto & [h \cup id_{\amalg_{i=1}^2 D^2}] \end{array}$$

and

$$\begin{array}{cccc} \eta: & \mathcal{M}_{g,2} & \to & \mathcal{M}_{g+1,0}. \\ & [h] & \mapsto & [h \cup id_{S^1 \times I}] \end{array}$$

Harer[3][4] shows that θ and η induce an isomorphism on the second homology classes when genus $g \geq 5$, so that $\tilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$ is a coboundary. Powell[11] proved that the first cohomology group $H_1(\mathcal{M}_{g,r}; \mathbf{Z})$ is trivial for $g \geq 3$ and $r \geq 0$, so by the universal coefficient theorem, it follows that the cobounding function of $\tilde{\tau}_g$ is unique.

In this paper we define a $\mathbf{QP}^1$-valued class function m on the mapping class group $\mathcal{M}_{g,2}$ in an explicit way by using information of the first homology group of a mapping torus of $[h] \in \mathcal{M}_{g,2}$, and prove that the sign of the function m cobounds the cocycle $\tilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$. Especially it turns out that the cocycle $\tilde{\tau}_g$ is coboundary for any $g \geq 0$.

In section 1, we construct a class function m, prove some properties of this function, and calculate the image of the function. In section 2, we prove that the sign of this function cobounds the difference $\tilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$. By the definition of the Meyer cocycle τ_g, $\tilde{\tau}_g(\varphi, \psi)$ is just the difference $\mathrm{Sign}\, E_{g+1}^{\eta(\varphi),\eta(\psi)} - \mathrm{Sign}\, E_g^{\theta(\varphi),\theta(\psi)}$, so that we calculate the difference by using the sign of the function m. Moreover we compute the other differences of signature $\mathrm{Sign}(E_{g,2}^{\varphi,\psi}) - \mathrm{Sign}(E_g^{\theta(\varphi),\theta(\psi)})$ and $\mathrm{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)}) - \mathrm{Sign}(E_{g,2}^{\varphi,\psi})$ by the function m.

1 Class function $m : \mathcal{M}_{g,2} \to \mathbf{QP}^1$

In this section we define the class function on the mapping class group $\mathcal{M}_{g,2}$ stated in Introduction and describe some properties of the function including the nontriviality.

For $[p:q]$, $[r:s] \in \mathbf{QP}^1$, we define an addition in $\mathbf{QP}^1$ by

$$[p:q] + [r:s] = \begin{cases} [pr : ps + qr], & \text{if} \quad [p:q] \neq [0:1] \text{ or } [r:s] \neq [0:1] \\ [0:1], & \text{if} \quad [p:q] = [r:s] = [0:1]. \end{cases}$$

The projective line $\mathbf{QP}^1$ forms an additive monoid under this operation with $[1:0]$ the zero element.

In this section, all (co)homology groups is with $\mathbf{Q}$ coefficients.

1.1 Construction of the class function

Before constructing the function, we prepare a fact about homology groups of compact 3-manifolds. Let Y be a compact oriented 3-manifold with boundary ∂Y and $i : \partial Y \hookrightarrow Y$ the inclusion map. Consider the commutative

diagram

$$\begin{array}{ccccc} H^1(Y) & \xrightarrow{i^*} & H^1(\partial Y) & \xrightarrow{\delta^*} & H^2(Y,\partial Y) \\ \downarrow{\scriptstyle \cap[Y]} & & \downarrow{\scriptstyle \cap[\partial Y]} & & \downarrow{\scriptstyle \cap[Y]} \\ H_2(Y,\partial Y) & \xrightarrow{\partial_*} & H_1(\partial Y) & \xrightarrow{i_*} & H_1(Y), \end{array}$$

where the upper and lower rows are the exact sequences of a pair $(Y, \partial Y)$, and the vertical maps are the cap products with the fundamental classes of Y and ∂Y. By the diagram and Poincaré Duality, it follows that the image of i^* is just its own annihilator with respect to the cup product of $H^1(\partial Y)$

$$\operatorname{Im} i^* = \operatorname{Ann}(\operatorname{Im} i^*).$$

In particular, we have

$$\dim \operatorname{Ker} i_* = \dim \operatorname{Im} i^* = \frac{1}{2} \dim H_1(\partial Y).$$

We define the mapping torus of $\varphi = [h] \in \mathcal{M}_{g,r}$ by

$$X^\varphi := \Sigma_{g,r} \times I/\sim, \quad (x,1) \sim (h(x),0),$$

and $\pi : X^\varphi \to I/\partial I = S^1$ by the projection $\pi([x,t]) = [t]$, where $[x,t] \in X^\varphi$ is the equivalent class of $(x,t) \in \Sigma_{g,r} \times I$, and $[t] \in I/\partial I = S^1$ the equivalent class of $t \in I$.

The diffeomorphism type of the mapping torus X^φ does not depend on the choice of the representative h. We fix the orientation on X^φ given by the product orientation on $\Sigma_{g,r} \times I$. Let $i_\varphi : \partial X^\varphi \hookrightarrow X^\varphi$ be the inclusion map. In this subsection we denote $\Sigma := \Sigma_{g,2}$, and if we fix $\varphi \in \mathcal{M}_{g,2}$, then we write simply $X := X^\varphi$ and $i := i_\varphi$. Let S_1 and S_2 be the two boundary components of Σ, and $[S_k]$ $(k = 1,2)$ the image under the inclusion homomorphism $H_1(S_k) \to H_1(\Sigma)$ of the fundamental homology class.

We consider Σ as a subspace of X by the embedding $\iota : \Sigma \hookrightarrow X$ $x \mapsto [x,0]$. We choose points $p_1 \in S_1$, $p_2 \in S_2$, and $p \in S^1$, and orientation-preserving homeomorphisms $\iota_1 : S^1 \to S_1$ and $\iota_2 : S^1 \to S_2$. We define singular cochains $f_k : I \to (S_1 \amalg S_2) \times S^1 = \partial X$ $(k = 1,2,3,4)$ by

$$f_1(t) = (\iota_1(t), p), \quad f_2(t) = (\iota_2(t), p), \quad f_3(t) = (p_1, t), \text{ and } \quad f_4(t) = (p_2, t), \text{ respectively.}$$

Let $e_k \in H_1(\partial X)$ be the homology class of f_k $(k = 1,2,3,4)$. Then the set $\{e_1, e_2, e_3, e_4\}$ forms a basis for $H_1(\partial X)$.

Now we describe the kernel of the homomorphism $i_* : H_1(\partial X) \to H_1(X)$. Since e_1 and e_2 lie in the kernel of $(\pi|_{\partial X})_*$ and $\pi_*(e_3) = \pi_*(e_4) = [S^1] \in H_1(S^1)$, we have

$$\operatorname{Ker}\, i_* \subset \operatorname{Ker}\,(\pi_* i_*) = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4).$$

By the definition of the map f_k, $(i \circ f_k)_*[S^1] = \iota_*[S_k]$, and so $i_*(e_1 + e_2) = \iota_*([S_1] + [S_2]) \in H_1(X)$. Since $S_1 \cup S_2$ is the boundary of Σ, we have $[S_1] + [S_2] = 0 \in H_1(\Sigma)$. Hence

$$\mathbf{Q}(e_1 + e_2) \subset \operatorname{Ker}\, i_*.$$

As we saw at the beginning of this subsection, $\dim \operatorname{Ker}\, i_* = \frac{1}{2} \dim H_1(\partial X) = 2$. It follows that $\operatorname{Ker} i_* = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + qe_1)$ for some p, $q \in \mathbf{Q}$. Now we can define a class function.

Definition 1.1. *For $\varphi \in \mathcal{M}_{g,2}$, we take $p, q \in \mathbf{Q}$ such that* $\operatorname{Ker} i_{\varphi *} = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + qe_1)$.

We define $m : \mathcal{M}_{g,2} \to \mathbf{QP^1}$ by $m(\varphi) = [p : q]$.

Lemma 1.2. *For $\varphi, \psi \in \mathcal{M}_{g,2}$,*

$$m(\psi\varphi\psi^{-1}) = m(\varphi).$$

Proof. Define $\Psi : X^\varphi \to X^{\psi\varphi\psi^{-1}}$ by $\Psi(x,t) = (\psi(x), t)$. Then the following diagram commutes

$$\begin{array}{ccc}
H_1(\partial X^\varphi) & \xrightarrow{i_{\varphi *}} & H_1(X^\varphi) \\
\Big\downarrow{\scriptstyle \Psi_*} & & \Big\downarrow{\scriptstyle \Psi_*} \\
H_1(\partial X^{\psi\varphi\psi^{-1}}) & \xrightarrow{i_{\psi\varphi\psi^{-1}*}} & H_1(X^{\psi\varphi\psi^{-1}}).
\end{array}$$

We can see from the diagram, Ψ_* gives the natural isomorphism between $\operatorname{Ker}(H_1(\partial X^\varphi) \to H_1(X^\varphi))$ and $\operatorname{Ker}(H_1(\partial X^{\psi\varphi\psi^{-1}}) \to H_1(X^{\psi\varphi\psi^{-1}}))$. Hence we have $m(\psi\varphi\psi^{-1}) = m(\varphi)$. $\square$

1.2 Some properties and the nontriviality of the class function

By the Serre spectral sequence, we have the exact sequence

$$0 \longrightarrow \operatorname{Coker}(\varphi_* - 1) \xrightarrow{\iota_*} H_1(X) \xrightarrow{\pi_*} H_1(S^1) \longrightarrow 0,$$

where $\operatorname{Coker}(\varphi_* - 1)$ is the cokernel of the homomorphism $\varphi_* - 1 : H_1(\Sigma) \to H_1(\Sigma)$.

Then we have a unique homomorphism $j_\varphi : \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) \to \operatorname{Coker}(\varphi_* - 1)$ such that the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) & \longrightarrow & H_1(\partial X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle j_\varphi} & & \Big\downarrow{\scriptstyle i_*} & & \Big\| & & \\
0 & \longrightarrow & \operatorname{Coker}(\varphi_* - 1) & \xrightarrow{\iota_*} & H_1(X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0
\end{array}$$

commutes. By the diagram, we have

$$\operatorname{Ker} i_* = \operatorname{Ker} j_\varphi\,, \text{ and}$$

$$j_\varphi(e_1) = -j_\varphi(e_2) = [S_1] \in \operatorname{Coker}(\varphi_* - 1).$$

Now we introduce a cochain $\omega_l \in C^1(\mathcal{M}_{g,2}; H_1(\Sigma))$ defined in [7]. On the fiber $\Sigma = \pi^{-1}(0) \subset X$, pick a path l such that $l(0) \in S_2$ and $l(1) \in S_1$. Define ω_l by

$$\omega_l(\varphi) := \varphi(l) - l \in H_1(\Sigma).$$

Then we have

Lemma 1.3.

$$j_\varphi(e_3 - e_4) = [\omega_l(\varphi)] \in \operatorname{Coker}(\varphi_* - 1).$$

Proof. Define a 2-chain $L : I \times I \to X$ by $L(s,t) = [l(s), t]$. Its boundary is given by $-i_*(e_3) + \varphi(l) + i_*(e_4) - l \in B_1(X)$. Hence,

$$i_*(e_3 - e_4) = \iota_*([\varphi(l) - l]) \in H_1(X)$$

Since ι_* is injective, the lemma follows. $\square$

From the lemma, we see the homology class $[\omega_l(\varphi)] \in \mathrm{Coker}(\varphi_* - 1)$ is independent of the choice of the path l. If $\omega_l(\varphi) = 0$, then $j_\varphi(e_3 - e_4) = 0$.

Remark 1.4. *If there exists a path l from a point in S_2 to a point in S_1 which has no common point with the support of a representative of $\varphi \in \mathcal{M}_{g,2}$, then $m(\varphi) = [1 : 0]$. In particular, $m(id) = [1 : 0]$, the zero element of the monoid $\mathbf{QP}^1$.*

At the beginning of this section, we defined the commutative monoid structure on $\mathbf{QP^1}$. So integral multiples of $m(\varphi)$ are well-defined.

Proposition 1.5. *If $\varphi \in \mathcal{M}_{g,2}$ and $k \in \mathbf{Z}$, then*

$$m(\varphi^k) = km(\varphi).$$

Proof. The proposition is trivial for $k = 0$ and $k = 1$. Assume $k \geq 2$.

Let $m(\varphi) = [p : q]$. By the definition of j_φ, $pj_\varphi(e_3 - e_4) = -q[S_1] \in \mathrm{Coker}(\varphi_* - 1)$. Hence, there exists $v \in H_1(\Sigma)$ such that

$$p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v \in H_1(\Sigma)$$

Apply φ^i $(i = 1, 2, \cdots k-1)$ to the both sides of the equation and sum over i. Then

$$\sum_{i=1}^{k-1} p(\varphi^{i+1}(l) - \varphi^i(l)) = -\sum_{i=1}^{k-1} \{[S_1] + (\varphi_*^{i+1}(v) - \varphi_*^i(v))\},$$

that is

$$p(\varphi^k(l) - l) = -kq[S_1] + (\varphi_*^k - 1)v.$$

Hence, $m(\varphi^k) = [p : kq] = km(\varphi)$ for $k > 0$.

By applying φ^{-1} to the equation $p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v$, we have

$$p[\varphi^{-1}(l) - l] = q[S_1] + (\varphi_*^{-1} - 1)v \in H_1(\Sigma).$$

Hence, $m(\varphi^{-1}) = [p : -q] = -m(\varphi)$. Since $m(\varphi^{-k}) = -m(\varphi^k) = -km(\varphi)$ for $k > 0$, the proposition follows for the case $k < 0$. $\square$

Now we compute the image of the function m. Especially we prove that m is nontrivial.

Proposition 1.6. *For $g \geq 1$, m is surjective. For $g = 0$,* $\mathrm{Im}(m) = [1 : \mathbf{Z}]$.

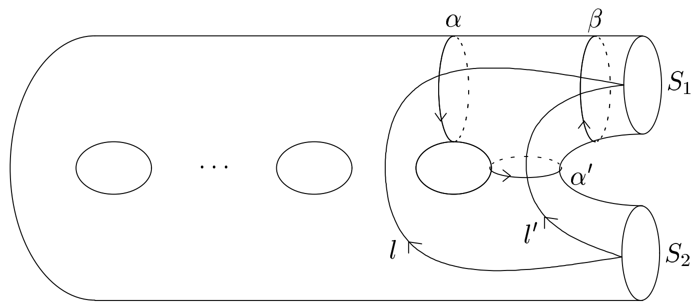

Figure 2:

Proof. Suppose $g \geq 1$. We choose oriented simple closed curves α, α', and β and paths l and l' as shown in Figure 2. We denote the Dehn twists along a simple closed curve $C \subset \Sigma$ by t_C, and the homology class of C by $[C]$. Then $[\alpha] + [\alpha'] + [\beta] = 0 \in H_1(\Sigma)$ since they bound a 2-chain. For $p \in \mathbf{Z}$, if we denote $\varphi := t_\alpha^p t_{\alpha'} t_\beta^{-1}$, then

$$\begin{aligned} j_\varphi((p+1)(e_3 - e_4)) &= \omega_l(\varphi) + p\omega_{l'}(\varphi) \\ &= (t_\alpha^p t_{\alpha'} t_\beta^{-1})(l) - l + p\{(t_\alpha^p t_{\alpha'} t_\beta^{-1})(l') - l'\} \\ &= p([\alpha] + [\alpha'] + [\beta]) + [\beta] = [\beta] = [S_1]. \end{aligned}$$

Hence, $j_\varphi((p+1)(e_3 - e_4) - e_1) = 0$, so that

$$m(\varphi) = [p+1 : -1].$$

By Proposition 2.5, we have

$$m(\varphi^{-q}) = -q[p+1 : -1] = \begin{cases} [p+1 : q], & \text{if } p \neq -1 \\ [0 : 1], & \text{if } p = -1. \end{cases} \qquad (q \in \mathbf{Z})$$

Since p and q can run over all integers, we see m is surjective for $g \geq 1$.

For $g = 0$, $\mathcal{M}_{0,2}$ is the infinite cyclic group generated by t_β. Since $m(t_\beta^{-q}) = [1 : q]$, we have $\mathrm{Im}(m) = [1 : \mathbf{Z}]$. $\square$

2 The difference of two Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$

In this section (co)homology groups are with $\mathbf{Z}$ coefficient unless specified.

Let $g \geq 0$ be a positive integer. In Introduction, we defined the homomorphisms $\eta : \mathcal{M}_{g,2} \to \mathcal{M}_{g+1,0}$ and $\theta : \mathcal{M}_{g,2} \to \mathcal{M}_g$ to be the induced maps by sewing a pair of disks and by sewing an annulus onto the surface $\Sigma_{g,2}$ along their boundaries respectively. We denote the Meyer cocycle on the mapping class group of genus g closed orientable surface $\mathcal{M}_g$ by $\tau_g \in Z^2(\mathcal{M}_g)$ and define $\tilde{\tau}_g \in Z^2(\mathcal{M}_{g,2})$ to be the difference between the Meyer cocycles

$$\tilde{\tau}_g := \eta^*\tau_{g+1} - \theta^*\tau_g.$$

Let $P := S^2 - \amalg_{i=1}^3 D^2$. In this section, we prove the main theorem and calculate the changes of signature associated with sewing a pair of trivial disk bundles $P \times \amalg_{i=1}^2 D^2$ and sewing an trivial annulus bundles $P \times (S^1 \times I)$ onto $\Sigma_{g,2}$ bundle on the pair of pants P along their boundaries. To state the main theorem, we define the sign of $[p : q] \in \mathbf{QP^1}$ by

$$\operatorname{sign}([p:q]) := \begin{cases} 1 & \text{if } pq > 0, \\ 0 & \text{if } pq = 0, \\ -1 & \text{if } pq < 0. \end{cases}$$

Theorem 2.1. *For $\varphi, \psi \in \mathcal{M}_{g,2}$, we define*

$$\tilde{\phi}_g(\varphi) := \operatorname{sign}(m(\varphi)).$$

Then $\tilde{\phi}_g$ cobounds the difference $\tilde{\tau}_g$ between the Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$

$$\begin{aligned} \tilde{\tau}_g(\varphi, \psi) &= \delta\tilde{\phi}_g(\varphi, \psi) \\ &= \operatorname{sign}(m(\varphi)) + \operatorname{sign}(m(\psi)) + \operatorname{sign}(m((\varphi\psi)^{-1})). \end{aligned}$$

Remark 2.2. *Let k be an integer. By Lemma 2.2 and Proposition 2.5, $\tilde{\phi}_g$ has the properties*

$$\begin{aligned} \tilde{\phi}_g(\psi\varphi\psi^{-1}) &= \tilde{\phi}_g(\varphi), \textit{and} \\ \tilde{\phi}_g(\varphi^k) &= \operatorname{sign}(k)\tilde{\phi}_g(\varphi) \end{aligned}$$

for any $g \geq 0$.

2.1 Proof of Main Theorem

In this subsection we prove Theorem 2.1.

In Introduction, we defined $E_{g,r}^{\varphi,\psi}$ as a $\Sigma_{g,r}$ bundle on the pair of pants P which has monodromies φ, ψ, and $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along α, β, and $\gamma \in \pi_1(P)$ respectively, and in Subsection 2.1, we defined $X_{g,r}^{\varphi}$ by the mapping torus of $\Sigma_{g,r} \times I/\sim$ where $(x,1) \sim (h(x),0)$ for $\varphi = [h] \in \mathcal{M}_{g,r}$.

We consider

$$E_{g+1}^{\eta(\varphi),\eta(\psi)} = E_{g,2}^{\varphi,\psi} \cup (-S^1 \times I \times P),$$

and

$$X_{g+1}^{\eta(\varphi)} = X_{g,2}^{\varphi} \cup (-S^1 \times I \times S^1).$$

Define

$$\begin{array}{rccc} G: & \partial D^2 \times I & \to & \{1\} \times S^1 \times I. \\ & (x,t) & \mapsto & (1, x, \frac{1+t}{3}) \end{array}$$

By the map G, we can glue $D^2 \times I$ to $I \times S^1 \times I$ as shown in figure 3. Glue $D^2 \times I \times P$ to $I \times E_{g+1}^{\eta(\varphi),\eta(\psi)} =$

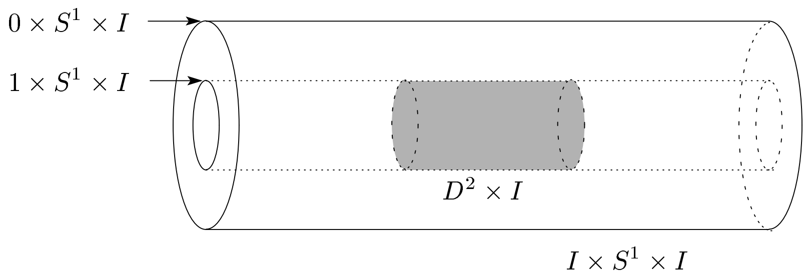

Figure 3: Gluing map G

$(I \times E_{g,2}^{\varphi,\psi}) \cup (-I \times S^1 \times I \times P)$ with the gluing map $G \times id_P : \partial D^2 \times I \times P \to \{1\} \times S^1 \times I \times P$. In the same way, glue $D^2 \times I \times S^1$ to $I \times X_{g+1}^{\eta(\varphi)} = (I \times X_{g,2}^{\varphi}) \cup (-I \times S^1 \times I \times S^1)$ with the gluing map $G \times id_{S^1} : \partial D^2 \times I \times S^1 \to \{1\} \times S^1 \times I \times S^1$. Denote

$$\tilde{E}^{\varphi,\psi} := (I \times E_{g+1}^{\eta(\varphi),\eta(\psi)}) \cup (D^2 \times I \times P), \text{ and } \tilde{X}^{\varphi} := (I \times X_{g+1}^{\eta(\varphi)}) \cup (D^2 \times I \times S^1).$$

To prove main theorem, it suffices to prove Lemma 2.3 and Lemma 2.4 below.

Lemma 2.3.

$$(\eta^*\tau_{g+1} - \theta^*\tau_g)(\varphi,\psi) = \operatorname{Sign}\tilde{X}^{\varphi} + \operatorname{Sign}\tilde{X}^{\psi} + \operatorname{Sign}\tilde{X}^{(\varphi\psi)^{-1}} \quad \textit{for } \varphi,\psi \in \mathcal{M}_{g,2},\ g \geq 0.$$

Lemma 2.4.

$$\operatorname{Sign}\tilde{X}^{\varphi} = \operatorname{sign}(m(\varphi)) \quad \textit{for } \varphi \in \mathcal{M}_{g,2},\ g \geq 0.$$

proof of Lemma 3.3. Note that

$$X^{\varphi} = \tilde{E}^{\varphi,\psi}|_{\partial D_1}.$$

Then we can see

$$\begin{aligned} \partial\tilde{E}^{\varphi,\psi} &= (\tilde{E}^{\varphi,\psi}|_{\partial D_1} \cup \tilde{E}^{\varphi,\psi}|_{\partial D_2} \cup \tilde{E}^{\varphi,\psi}|_{\partial D_3}) \cup E_g^{\theta(\varphi),\theta(\psi)} \cup -E_{g+1}^{\eta(\varphi),\eta(\psi)} \\ &= (\tilde{X}^{\varphi} \cup \tilde{X}^{\psi} \cup \tilde{X}^{(\psi\varphi)^{-1}}) \cup E_g^{\theta(\varphi),\theta(\psi)} \cup -E_{g+1}^{\eta(\varphi),\eta(\psi)}. \end{aligned}$$

By Novikov Additivity, the fact $\operatorname{Sign}\partial\tilde{E}^{\varphi,\psi}=0$ implies

$$\operatorname{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)})-\operatorname{Sign}(E_{g}^{\theta(\varphi),\theta(\psi)})=\operatorname{Sign}\tilde{X}^{\varphi}+\operatorname{Sign}\tilde{X}^{\psi}+\operatorname{Sign}\tilde{X}^{(\psi\varphi)^{-1}}.$$

Notice that $\tilde{X}^{(\psi\varphi)^{-1}}$ is diffeomorphic to $\tilde{X}^{(\varphi\psi)^{-1}}$, so that $\operatorname{Sign}\tilde{X}^{(\psi\varphi)^{-1}}=\operatorname{Sign}\tilde{X}^{(\varphi\psi)^{-1}}$. By the definition of the Meyer cocycle, we have

$$\operatorname{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)})=\eta^{*}\tau_{g+1}(\varphi,\psi),\text{ and }\ \operatorname{Sign}(E_{g}^{\theta(\varphi),\theta(\psi)})=\theta^{*}\tau_{g}(\varphi,\psi).$$

Define $\tilde{\phi}(\varphi)=\operatorname{Sign}(\tilde{X}^{\varphi})$, then we have $\delta\tilde{\phi}=\eta^{*}\tau_{g+1}-\theta^{*}\tau_{g}$. We get the cobounding function $\tilde{\phi}$. ∎

proof of Lemma 3.4. Write simply $X:=X_{g+1}^{\eta(\varphi)}$, $X':=X_{g,2}^{\varphi}$, and $Y:=\tilde{X}^{\varphi}=(I\times X)\cup(D^{2}\times I\times S^{1})$.

For $i=0,1$, define

$$\begin{array}{rccccc} j_i: & X & \to & I\times X & \hookrightarrow & Y, \\ & x & \mapsto & (i,x) & & \end{array}$$

where $I\times X\hookrightarrow Y$ is a natural embedding. We will prove there is a exact sequence

$$H_2(X')\xrightarrow{j_{0*}=j_{1*}}H_2(Y)\longrightarrow\operatorname{Ker}(H_1(\partial X')\to H_1(X'))\longrightarrow 0.$$

Define $Y_1:=I\times X'$ and $Y_2:=(I\times S^{1}\times I\times S^{1})\cup(D^{2}\times I\times S^{1})\subset Y$, then

$$Y_1\simeq X',Y_2\simeq S^{1},Y_1\cap Y_2\simeq\partial X'=(S_1\amalg S_2)\times S^{1}.$$

By the Mayer-Vietoris exact sequence, we have

$$\begin{array}{ccccccc} H_2(Y_1)\oplus H_2(Y_2) & \to & H_2(Y) & \to & H_1(Y_1\cap Y_2) & \to & H_1(Y_1)\oplus H_1(Y_2) \quad \text{(exact)}. \\ \wr\Vert & & & & \wr\Vert & & \wr\Vert \\ H_2(X')\oplus 0 & & & & H_1(\partial X') & & H_1(X')\oplus H_1(S^1) \end{array}$$

Denote the map $H_1(\partial X')\to H_1(X')\oplus H_1(S^1)$ in the above diagram by h. the projection $H_1(\partial X')\to H_1(S^1)$ to the second entry of h is the composite of inclusion homomorphism $H_1(\partial X')\to H_1(X')$ and $\pi_*:H_1(X')\to H_1(S^1)$. Therefore,

$$\operatorname{Ker}(H_1(\partial X')\to H_1(X')\oplus H_1(S^1))=\operatorname{Ker}(H_1(\partial X')\to H_1(X')).$$

So the sequence is exact.

Next we construct the splitting $H_2(Y;\mathbf{Q})=j_{i*}H_2(X';\mathbf{Q})\oplus\operatorname{Ker}(H_1(\partial X';\mathbf{Q})\to H_1(X';\mathbf{Q}))$. Note that there exist p, $q\in\mathbf{Q}$ such that

$$\operatorname{Ker}(H_1(\partial X';\mathbf{Q})\to H_1(X';\mathbf{Q}))=\mathbf{Q}(e_1+e_2)\oplus\mathbf{Q}\{p(e_3-e_4)+qe_1\}$$

as in section 1. To construct the splitting, we choose elements of inverse images of e_1+e_2, $p(e_3-e_4)+qe_1$ under $H_2(Y)\to H_1(\partial X')$. Define $\iota_Y:\Sigma_{g+1}\to Y$ by

$$\begin{array}{ccccccc} \Sigma_{g+1} & \to & X & \to & I\times X & \hookrightarrow & Y, \\ x & \mapsto & (x,0) & \mapsto & (0,x,0) & & \end{array}$$

then we have

$$\begin{array}{ccccc} H_2(\tilde{X}) & \to & H_1(Y_1 \cap Y_2) & \to & H_1(\partial X'), \\ \iota_{Y*}[\Sigma_g] & \mapsto & \partial_* \iota_{Y*}[\Sigma_g] & \to & e_1 + e_2 \end{array}$$

so we choose $\iota_{Y*}[\Sigma_g]$ as an element of the inverse image of $e_1 + e_2$.

Next, we choose an element of the inverse image of $p(e_3-e_4)+qe_1$. Since $p(e_3-e_4)+qe_1 \in \mathrm{Ker}(H_1(\partial X'; \mathbf{Q}) \to H_1(X'; \mathbf{Q}))$, there exists a singular 2-cochain $s \in C_2(X'; \mathbf{Q})$ such that

$$\partial s = p(f_3 - f_4) + q f_1 \in B_1(X'; \mathbf{Q}).$$

For $i = 0, 1$, define $s'_{0i} : I \times S^1 \to I \times S^1 \times I \times S^1 \hookrightarrow Y_2$ by $s'_{0i}(s,t) = (i, 0, s, t)$. then

$$[\partial s'_{0i}] = [j_i f_3 - j_i f_4] \in H_1(Y_1 \cap Y_2; \mathbf{Q}).$$

Define $s'_{1i} : D^2 \to (-I \times S^1 \times I \times S^1) \cup (D^2 \times I \times S^1) \subset Y$ as shown in Figure 4 by

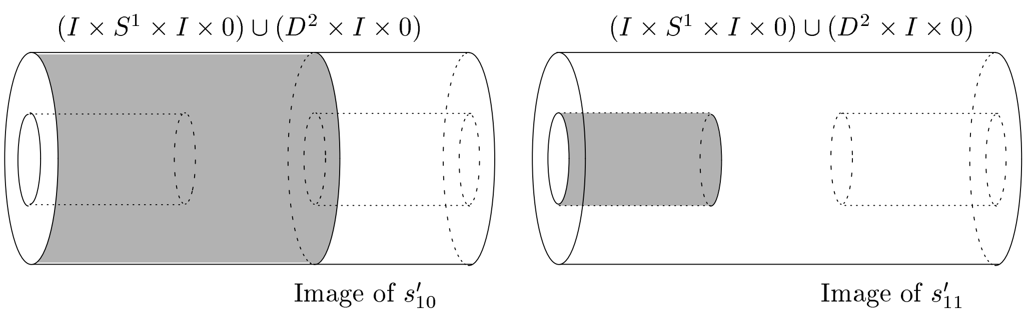

Figure 4: Images of s'_{10} and $s'_{11} \subset (I \times S^1 \times I \times 0) \cup (D^2 \times I \times 0) \subset Y$

$$s'_{10}(x) = \begin{cases} (6x, 1, 0) & \in D^2 \times I \times S^1 \quad (||x|| \le \frac{1}{6}), \\ (2 - 6||x||, \frac{x}{||x||}, \frac{2}{3}, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{1}{6} \le ||x|| \le \frac{1}{3}), \\ (0, 1 - ||x||, \frac{x}{||x||}, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{1}{3} \le ||x|| \le 1), \end{cases}$$

$$s'_{11}(x,t) = \begin{cases} (\frac{3}{2}x, 0, 0) & \in D^2 \times I \times S^1 \quad (||x|| \le \frac{2}{3}), \\ (1, \frac{x}{||x||}, 1 - ||x||, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{2}{3} \le ||x|| \le 1). \end{cases}$$

Then, we have $[\partial s'_{1i}] = [j_i f_1] \in H_1(Y_1 \cap Y_2; \mathbf{Q})$.

Define $s'_i = p s'_{0i} + q s'_{1i}$, then it follows that

$$[\partial s'_i] = [j_i(p(f_3 - f_4) + q f_1)] \in H_1(Y_1 \cap Y_2; \mathbf{Q}),$$

so that we have $[\partial(j_i s - s'_i)] = 0 \in H_1(Y_1 \cap Y_2; \mathbf{Q})$.

We see

$$\begin{array}{ccccc} H_2(Y; \mathbf{Q}) & \to & H_1(Y_1 \cap Y_2; \mathbf{Q}) & \to & H_1(\partial X'; \mathbf{Q}), \\ [j_i s - s'_i] & \mapsto & \partial_*[j_i s - s'_i] & \mapsto & p(e_3 - e_4) + q e_1 \end{array}$$

so that we can choose $[j_i s - s'_i]$ as an element of the inverse image of $p(e_3 - e_4) + qe_1$.

Now we calculate the intersection form of $H_2(Y; \mathbf{Q})$. Define $X''_1 = j_1(X) \cup (D^2 \times 0 \times S^1) \subset (I \times X) \cup (D^2 \times I \times S^1) \subset Y$, then X''_1 is deformation retract of Y. Hence, every element of $H_2(Y; \mathbf{Q})$ is represented by a cocycle in X''_1. Therefore, a cohomology class is included in the annihilator of intersection form in $H_2(Y; \mathbf{Q})$ if it is represented by a cocycle which have no common point with X''_1. We see

$$j_0(X') \cap X''_1 = \emptyset, \text{ and } \ \iota_Y(\Sigma_{g+1}) \cap X''_1 = \emptyset,$$

so that $\mathbf{Q}(e_1 + e_2)$ and $j_{0*}H_2(X'; \mathbf{Q})$ are included in the annihilator of intersection form in $H_2(Y; \mathbf{Q})$.

To describe the signature of Y, it suffices to calculate the self-intersection number of $[j_i s - s'_i] = p(e_3 - e_4) + qe_1$. The cocycle $j_i s - s'_i$ satisfies

$$\begin{gathered}
\mathrm{Im}(j_0 s) \cap (\mathrm{Im}(j_1 s) \cup \mathrm{Im}(s'_{01}) \cup \mathrm{Im}(s'_{11})) = \emptyset \\
\mathrm{Im}(s'_{00}) \cap (\mathrm{Im}(j_1 s) \cup \mathrm{Im}(s'_{01})) = \emptyset \\
\mathrm{Im}(s'_{10}) \cap (\mathrm{Im}(j_1 s) \cup \mathrm{Im}(s'_{01}) \cup \mathrm{Im}(s'_{11})) = \emptyset,
\end{gathered}$$

so that

$$\begin{aligned}
(j_0 s - s'_0) \cdot (j_1 s - s'_1) &= (j_0 s - (ps'_{00} + qs'_{10})) \cdot (j_1 s - (ps'_{01} + qs'_{11})) \\
&= ps'_{00} \cdot qs'_{11}.
\end{aligned}$$

We can see s'_{00} and s'_{11} intersect only once positively. Hence, $\mathrm{Sign}(Y) = \mathrm{Sign}(pq) = \mathrm{Sign}(m(\varphi))$. □

2.2 Wall's Non-additivity Formula

Wall derives the Novikov additivity for a more general case: two compact oriented smooth $4k$-manfolds are glued along a common submanifolds, which itself have boundary, of the boundaries of the original manifolds.

We will give the specific case of his formula for $k = 1$:

Let Z be a closed oriented smooth 2-manifold, X_-, X_0, X_+ compact oriented smooth 3-manifolds with the boundaries $\partial X_- = \partial X_0 = \partial X_+ = Z$, and Y_-, Y_+ compact oriented smooth 4-manifolds with the boundaries $\partial Y_- = X_- \cup_Z (-X_0)$, $\partial Y_+ = X_0 \cup_Z (-X_+)$. Here we denote by $M \cup_B (-N)$ the union of two manifolds M and N glued by orientation reversing diffeomorphism of their common boundaries $\partial M = \partial N = B$. Let $Y = Y_- \cup_{X_0} Y_+$ be the union of Y_- and Y_+ glued along submanifolds X_0 of their boundaries. Suppose Y is oriented by the induced orientation of Y_- and Y_+.

Write $V = H_1(Z; \mathbf{R})$; let A, B, and C be the kernels of the maps on first homology induce by the inclusions of Z in X_-, X_0 and X_+ respectively.

We define

$$W := \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)},$$

and a bilinear form Ψ by

$$\begin{array}{cccccc}
\Psi : & W & \times & W & \to & \mathbf{R}. \\
 & (b & , & b') & \mapsto & b \cdot c'
\end{array}$$

Here c' is a element which satisfies $a' + b' + c' = 0$, and $b \cdot c'$ denote the intersection product of b and c'.

Then Ψ is independent of c' and well-defined on W. Denote the signature of the bilinear form Ψ by $\mathrm{Sign}(V; BCA)$ and the signature of the compact oriented 4-manifold M by $\mathrm{Sign}\, M$. We are now ready to state the formula.

Theorem 2.5 (Wall[12])**.** $\mathrm{Sign}\, Y = \mathrm{Sign}\, Y_- + \mathrm{Sign}\, Y_+ - \mathrm{Sign}(V; BCA)$.

2.3 The differences of signature $\mathrm{Sign}\, E_g - \mathrm{Sign}\, E_{g,2}$ and $\mathrm{Sign}\, E_{g+1} - \mathrm{Sign}\, E_{g,2}$

In this subsection, we calculate the difference of signature associated with sewing the trivial Disk bundles onto the $\Sigma_{g,2}$ bundle.

In Introduction, we defined $E_{g,r}^{\varphi,\psi}$ as a oriented $\Sigma_{g,r}$ bundle on P which has monodromies $\varphi, \psi, (\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha, \beta, \gamma \in \pi_1(P)$. If we fix $\varphi, \psi \in \mathcal{M}_{g,2}$, we denote simply

$$E_{g,2} := E_{g,2}^{\varphi,\psi}, \quad E_g := E_g^{\theta(\varphi),\theta(\psi)}, \text{ and } \quad E_{g+1} := E_{g+1}^{\eta(\varphi),\eta(\psi)} \quad (g \geq 0).$$

Proposition 2.6. $\mathrm{Sign}(E_g) - \mathrm{Sign}(E_{g,2}) = -\mathrm{Sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})) \quad (g \geq 0)$

Proof. E_g is the union of $E_{g,2}$ and $E_D := (D^2 \amalg D^2) \times P$ glued along their boundaries. Using Non-additivity formula Theorem 2.5, we calculate $\mathrm{Sign}(E_g) - \mathrm{Sign}(E_{g,2})$.

Define Y_-, Y_+, X_-, X_0, X_+, and Z by

$$\begin{gathered} Y_- := (\amalg_{j=1}^2 D^2) \times P, \quad Y_+ := E_{g,2}, \\ X_- := (\amalg_{j=1}^2 D^2) \times \partial P, \quad X_+ := E_{g,2}|_{\partial P}, \quad X_0 := (\amalg_{j=1}^2 \partial D^2) \times P, \\ \text{and } Z := (\amalg_{j=1}^2 \partial D^2) \times \partial P, \quad \text{respectively.} \end{gathered}$$

Here, by the notation stated in subsection 1.1,

$$X_+ = E_{g,2}|_{\partial P} \cong X^\varphi \amalg X^\psi \amalg X^{(\psi\varphi)^{-1}}, \quad Z \cong \partial X^\varphi \amalg \partial X^\psi \amalg \partial X^{(\psi\varphi)^{-1}}.$$

Define V, A, B, and C as stated in subsection 3.1.

Since $X^\varphi = X^\psi = X^{(\psi\varphi)^{-1}} = S^1 \times S^1$, we can choose the base of $H_1(\partial X^\varphi; \mathbf{R})$, $H_1(\partial X^\psi; \mathbf{R})$, and $H_1(\partial X^{(\psi\varphi)^{-1}}; \mathbf{R})$ as in section 1.1. Denote their base by $\{e_{11}, e_{12}, e_{13}, e_{14}\}$, $\{e_{21}, e_{22}, e_{23}, e_{24}\}$, $\{e_{31}, e_{32}, e_{33}, e_{34}\}$ respectively.

Since $Z = \partial X^\varphi \amalg \partial X^\psi \amalg \partial X^{(\psi\varphi)^{-1}}$, we think of e_{ij} as an element of $H_1(Z; \mathbf{R})$.

Denote $m(\varphi) = [a_1 : b_1]$, $m(\psi) = [a_2 : b_2]$, and $m((\psi\varphi)^{-1}) = [a_3 : b_3]$ respectively, then

$$
\begin{aligned}
V &= H_1(Z, \mathbf{R}) = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{4} \mathbf{R} e_{ij}, \\
A &= \mathbf{R}e_{11} \oplus \mathbf{R}e_{21} \oplus \mathbf{R}e_{31} \oplus \mathbf{R}e_{12} \oplus \mathbf{R}e_{22} \oplus \mathbf{R}e_{32}, \\
B &= \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{12} - e_{32}) \\
&\quad \oplus \mathbf{R}(e_{13} + e_{23} + e_{33}) \oplus \mathbf{R}(e_{14} + e_{24} + e_{34}), \\
C &= \bigoplus_{i=1}^{3} \begin{cases} \mathbf{R}(e_{i1} + e_{i2}) \oplus \mathbf{R}(e_{i3} - e_{i4} + m_i e_{i1}) & \text{if } a_i \neq 0 \\ \mathbf{R}e_{i1} \oplus \mathbf{R}e_{i2} & \text{if } a_i = 0. \end{cases} \qquad \text{Here we denote } m_i := \frac{b_i}{a_i}.
\end{aligned}
$$

Hence,

$$
B \cap A = \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}),
$$

$$
B \cap C = \begin{cases}
\mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) & \text{if } a_i \neq 0 \quad \text{for } i = 1, 2, 3 \\
\oplus \mathbf{R}(e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34} + m_1 e_{11} + m_2 e_{21} + m_3 e_{31}) & \text{and } m_1 + m_2 + m_3 = 0, \\[1ex]
\mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) & \text{if } a_i \neq 0 \quad \text{for } i = 1, 2, 3 \\
\oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) & \text{and } m_1 + m_2 + m_3 \neq 0, \\[1ex]
\mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) & \text{if } a_1 = 0, a_2 \neq 0, a_3 \neq 0, \\[1ex]
\mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) & \text{if } a_1 = a_2 = 0, a_3 \neq 0, \\[1ex]
\mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}) & \text{if } a_i = 0 \quad \text{for } i = 1, 2, 3,
\end{cases}
$$

$$
B \cap (C + A) = \begin{cases}
\mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}) & \\
\oplus \mathbf{R}(e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34}) & \text{if } a_i \neq 0 \quad \text{for } i = 1, 2, 3, \\[1ex]
\mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) & \\
\oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}), & \text{otherwise.}
\end{cases}
$$

By computing the signature of Ψ, we have

$$
\mathrm{Sign}(V; BCA) = \begin{cases} \mathrm{Sign}(m_1 + m_2 + m_3) & \text{if } a_i \neq 0 \quad \text{for } i = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}
$$

Hence,

$$
\begin{aligned}
\mathrm{Sign}(V; BCA) &= \mathrm{Sign}(m(\varphi) + m(\psi) + m((\psi\varphi)^{-1})) \\
&= \mathrm{Sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).
\end{aligned}
$$

By Non-additivity formula, we have

$$\mathrm{Sign}(E_g) = \mathrm{Sign}(E_D) + \mathrm{Sign}(E_{g,2}) - \mathrm{Sign}(V; BCA).$$

Since E_D is a trivial bundle $(D^2 \amalg D^2) \times P$, we have $\mathrm{Sign}(E_D) = 0$.

This completes the proof of the proposition. □

By the theorem and Proposition 2.6, we can calculate the difference of signature $\mathrm{Sign}(E_g) - \mathrm{Sign}(E_{g,2})$.

Corollary 2.7. *For $g \geq 0$,*

$$\mathrm{Sign}(E_{g+1}) - \mathrm{Sign}(E_{g,2}) = \mathrm{Sign}(m(a)) + \mathrm{Sign}(m(b)) + \mathrm{Sign}(m((ab)^{-1})) \\ - \mathrm{Sign}(m(a) + m(b) + m((ab)^{-1})).$$

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